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The Math of the Golden Ratio

Posted on August 4, 2014 by mugadonna

Mathematics

Golden ratio conjugate

The negative root of the quadratic equation for **φ** (the "conjugate root") is

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$$-\frac{1}{\varphi} = 1 - \varphi = \frac{1-\sqrt{5}}{2} = -0.6180339887\ldots.$$

The absolute value of this quantity (≈ 0.618) corresponds to the length ratio taken in reverse order (shorter segment length over longer segment length, *b/a*), and is sometimes referred to as the *golden ratio conjugate*.[10] It is denoted here by the capital Phi (**Φ**):

$$\Phi = \frac{1}{\varphi} = \varphi^{-1} = 0.6180339887\ldots.$$

Alternatively, **Φ** can be expressed as

$$\Phi = \varphi - 1 = 1.6180339887\ldots - 1 = 0.6180339887\ldots.$$

This illustrates the unique property of the golden ratio among positive numbers, that

$$\frac{1}{\varphi} = \varphi - 1,$$

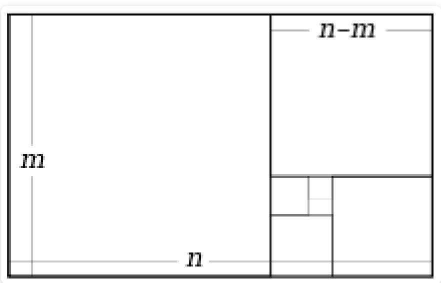
or its inverse:

$$\frac{1}{\Phi} = \Phi + 1.$$

This means 0.61803…:1 = 1:1.61803….

Short proofs of irrationality

Contradiction from an expression in lowest terms



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If φ were rational, then it would be the ratio of sides of a rectangle with integer sides. But it is also a ratio of sides, which are also integers, of the smaller rectangle obtained by deleting a square. The sequence of decreasing integer side lengths formed by deleting squares cannot be continued indefinitely, so φ cannot be rational.

Recall that:

the whole is the longer part plus the shorter part;

the whole is to the longer part as the longer part is to the shorter part.

If we call the whole *n* and the longer part *m*, then the second statement above becomes

n is to *m* as *m* is to *n* − *m*,

or, algebraically

$$\frac{n}{m} = \frac{m}{n-m}. \qquad (*)$$

To say that *φ* is rational means that *φ* is a fraction *n*/*m* where *n* and *m* are integers. We may take *n*/*m* to be in lowest terms and *n* and *m* to be positive. But if *n*/*m* is in lowest terms, then the identity labeled (*) above says *m*/(*n* − *m*) is in still lower terms. That is a contradiction that follows from the assumption that *φ* is rational.

Derivation from irrationality of √5

Another short proof—perhaps more commonly known—of the irrationality of the golden ratio makes use of the closure of rational numbers under addition and multiplication. If $\frac{1+\sqrt{5}}{2}$ is rational, then $2\left(\frac{1+\sqrt{5}}{2}\right) - 1 = \sqrt{5}$ is also rational, which is a contradiction if it is already known that the square root of a non-square natural number is irrational.

Pacifism

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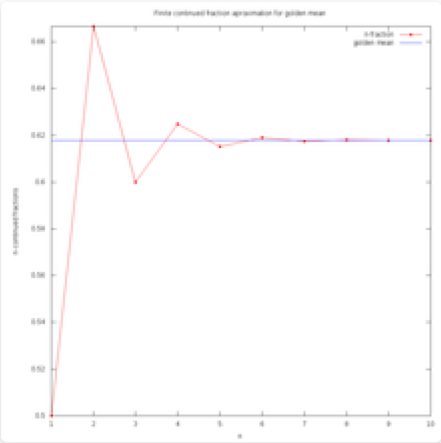
Alternative forms



Approximations to the reciprocal golden ratio by finite continued fractions, or ratios of Fibonacci numbers

The formula $\varphi = 1 + 1/\varphi$ can be expanded recursively to obtain a continued fraction for the golden ratio:[64]

$$\varphi = [1; 1, 1, 1, \dots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

and its reciprocal:

$$\varphi^{-1} = [0; 1, 1, 1, \dots] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

The convergents of these continued fractions (1/1, 2/1, 3/2, 5/3, 8/5, 13/8, …, or 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, …) are ratios of successive Fibonacci numbers.

The equation $\varphi^2 = 1 + \varphi$ likewise produces the continued square root, or infinite surd, form:

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

An infinite series can be derived to express phi:[65]

$$\varphi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}(2n+1)!}{(n+2)!n!4^{(2n+3)}}.$$

Also:

$$\varphi = 1 + 2\sin(\pi/10) = 1 + 2\sin 18^\circ$$

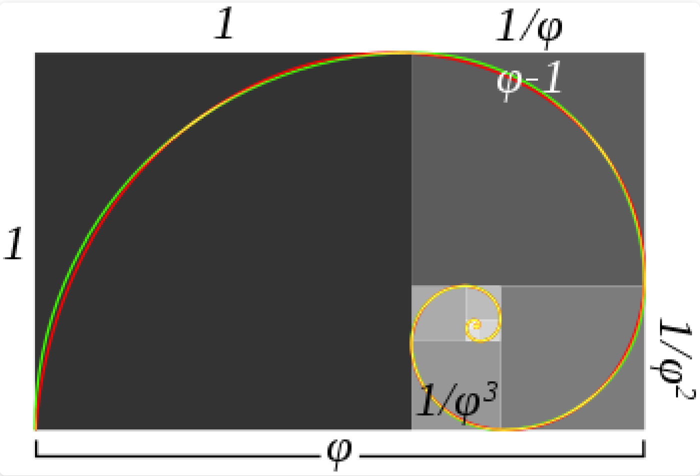
$$\varphi = \frac{1}{2}\csc(\pi/10) = \frac{1}{2}\csc 18^\circ$$

$$\varphi = 2\cos(\pi/5) = 2\cos 36^\circ$$

$$\varphi = 2\sin(3\pi/10) = 2\sin 54^\circ.$$

These correspond to the fact that the length of the diagonal of a regular pentagon is φ times the length of its side, and similar relations in a pentagram.

Geometry

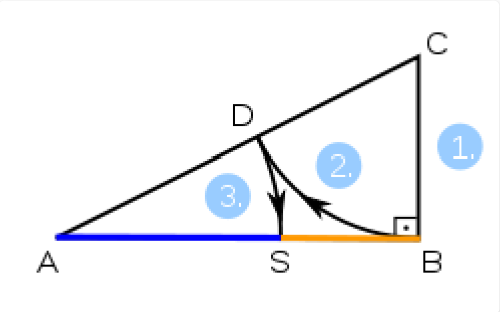


Approximate and true golden spirals. The green spiral is made from quarter-circles tangent to the interior of each square, while the red spiral is a Golden Spiral, a special type of logarithmic spiral. Overlapping portions appear yellow. The length of the side of one square divided by that of the next smaller square is the golden ratio.

The number φ turns up frequently in geometry, particularly in figures with pentagonal symmetry. The length of a regular pentagon's diagonal is φ times its side. The vertices of a regular icosahedron are those of three mutually orthogonal golden rectangles.

There is no known general algorithm to arrange a given number of nodes evenly on a sphere, for any of several definitions of even distribution (see, for example, *Thomson problem*). However, a useful approximation results from dividing the sphere into parallel bands of equal surface area and placing one node in each band at longitudes spaced by a golden section of the circle, i.e. 360°/φ ≅ 222.5°. This method was used to arrange the 1500 mirrors of the student-participatory satellite Starshine-3.[66]

Dividing a line segment



Dividing a line segment according to the golden ratio

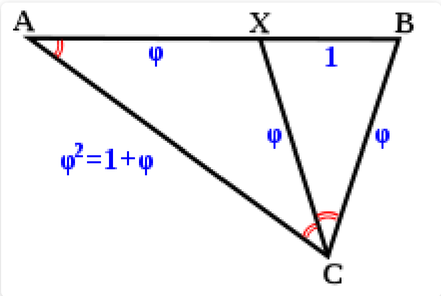
The following algorithm produces a geometric construction that divides a line segment into two line segments where the ratio of the longer to the shorter line segment is the golden ratio:

1. Having a line segment AB, construct a perpendicular BC at point B, with BC half the length of AB. Draw the

hypotenuse AC.

2. Draw an arc with center C and radius BC. This arc intersects the hypotenuse AC at point D.
3. Draw an arc with center A and radius AD. This arc intersects the original line segment AB at point S. Point S divides the original segment AB into line segments AS and SB with lengths in the golden ratio.

Golden triangle, pentagon and pentagram



Golden triangle

Golden triangle

The golden triangle can be characterized as an isosceles triangle ABC with the property that bisecting the angle C produces a new triangle CXB which is a similar triangle to the original.

If angle BCX = α, then XCA = α because of the bisection, and CAB = α because of the similar triangles; ABC = 2α from the original isosceles symmetry, and BXC = 2α by similarity. The angles in a triangle add up to 180°, so 5α = 180, giving α = 36°. So the angles of the golden triangle are thus 36°-72°-72°. The angles of the remaining obtuse isosceles triangle AXC (sometimes called the golden gnomon) are 36°-36°-108°.

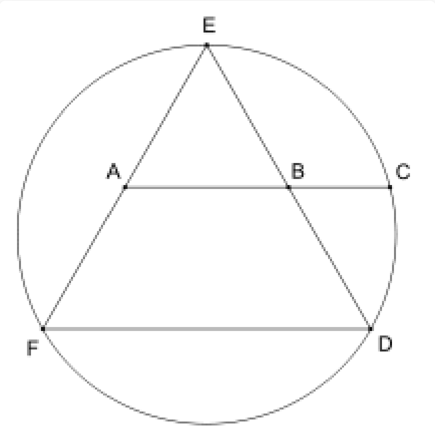
Suppose XB has length 1, and we call BC length φ. Because of the isosceles triangles XC=XA and BC=XC, so these are also length φ. Length AC = AB, therefore equals φ + 1. But triangle ABC is similar to triangle CXB, so AC/BC = BC/BX, and so AC also equals φ^2. Thus $\varphi^2 = \varphi + 1$, confirming that φ is indeed the golden ratio.

Similarly, the ratio of the area of the larger triangle AXC to the smaller CXB is equal to φ, while the inverse ratio is φ − 1.

Pentagon

In a regular pentagon the ratio between a side and a diagonal is Φ (i.e. 1/φ), while intersecting diagonals section each other in the golden ratio.[8]

Odom's construction

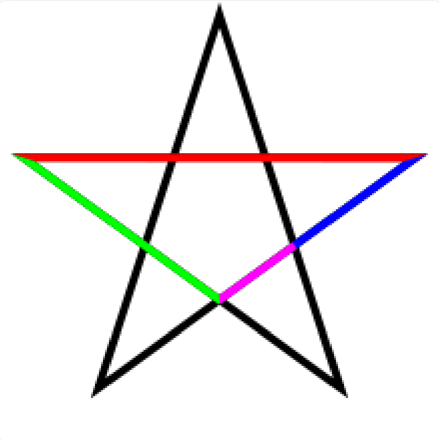


Let A and B be midpoints of the sides EF and ED of an equilateral triangle DEF. Extend AB to meet the circumcircle of DEF at C.

$$\frac{|AB|}{|BC|} = \frac{|AC|}{|AB|} = \phi$$

George Odom has given a remarkably simple construction for *φ* involving an equilateral triangle: if an equilateral triangle is inscribed in a circle and the line segment joining the midpoints of two sides is produced to intersect the circle in either of two points, then these three points are in golden proportion. This result is a straightforward consequence of the intersecting chords theorem and can be used to construct a regular pentagon, a construction that attracted the attention of the noted Canadian geometer H. S. M. Coxeter who published it in Odom's name as a diagram in the *American Mathematical Monthly* accompanied by the single word “Behold!” [67]

Pentagram

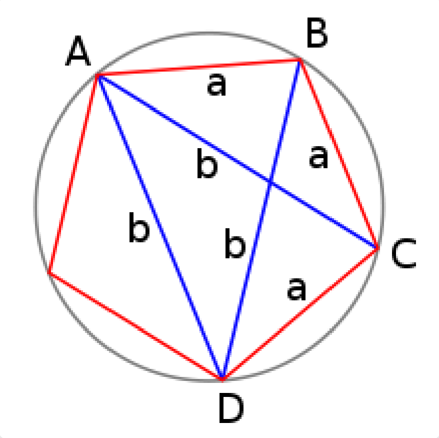


A pentagram colored to distinguish its line segments of different lengths. The four lengths are in golden ratio to one another.

The golden ratio plays an important role in the geometry of pentagrams. Each intersection of edges sections other edges in the golden ratio. Also, the ratio of the length of the shorter segment to the segment bounded by the two intersecting edges (a side of the pentagon in the pentagram’s center) is φ, as the four-color illustration shows.

The pentagram includes ten isosceles triangles: five acute and five obtuse isosceles triangles. In all of them, the ratio of the longer side to the shorter side is φ. The acute triangles are golden triangles. The obtuse isosceles triangles are golden gnomons.

Ptolemy’s theorem



The golden ratio in a regular pentagon can be computed using Ptolemy's theorem.

The golden ratio properties of a regular pentagon can be confirmed by applying Ptolemy's theorem to the quadrilateral formed by removing one of its vertices. If the quadrilateral's long edge and diagonals are *b*, and short edges are *a*, then Ptolemy's theorem gives $b^2 = a^2 + ab$ which yields

$$\frac{b}{a} = \frac{1+\sqrt{5}}{2}.$$

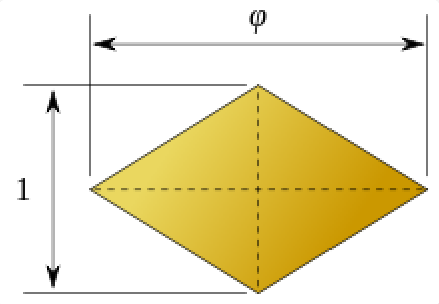
Scalenity of triangles

Consider a triangle with sides of lengths *a*, *b*, and *c* in decreasing order. Define the "scalenity" of the triangle to be the smaller of the two ratios a/b and b/c. The scalenity is always less than φ and can be made as close as desired to φ.[68]

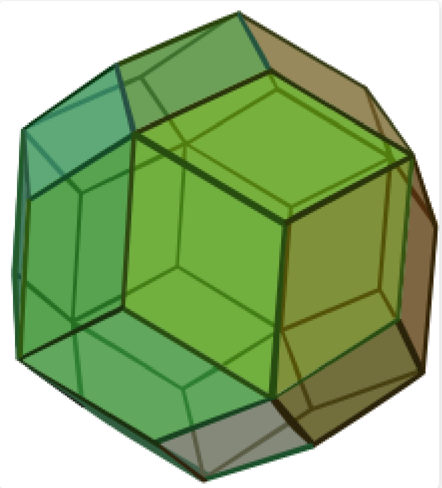
Triangle whose sides form a geometric progression

If the side lengths of a triangle form a geometric progression and are in the ratio $1 : r : r^2$, where *r* is the common ratio, then *r* must lie in the range $\varphi-1 < r < \varphi$, which is a consequence of the triangle inequality (the sum of any two sides of a triangle must be strictly bigger than the length of the third side). If $r = \varphi$ then the shorter two sides are 1 and φ but their sum is φ^2, thus $r < \varphi$. A similar calculation shows that $r > \varphi-1$. A triangle whose sides are in the ratio $1 : \sqrt{\varphi} : \varphi$ is a right triangle (because $1 + \varphi = \varphi^2$) known as a Kepler triangle.[69]

Golden triangle, rhombus, and rhombic triacontahedron



One of the rhombic triacontahedron's rhombi



All of the faces of the rhombic triacontahedron are golden rhombi

A golden rhombus is a rhombus whose diagonals are in the golden ratio. The rhombic triacontahedron is a convex polytope that has a very special property: all of its faces are golden rhombi. In the rhombic triacontahedron the dihedral angle between any two adjacent rhombi is 144°, which is twice the isosceles angle of a golden triangle and four times its most acute angle.[70]

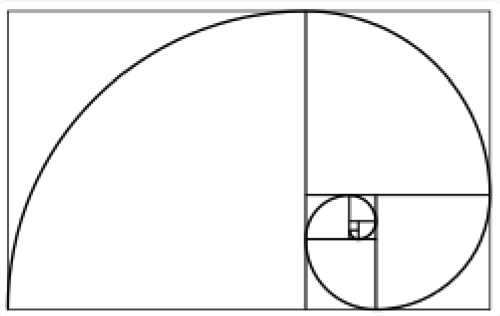
Relationship to Fibonacci sequence

The mathematics of the golden ratio and of the Fibonacci sequence are intimately interconnected. The Fibonacci sequence is:

> 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ….

The closed-form expression (known as Binet's formula, even though it was already known by Abraham de Moivre) for the Fibonacci sequence involves the golden ratio:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$



A Fibonacci spiral which approximates the golden spiral, using Fibonacci sequence square sizes up to 34.

The golden ratio is the limit of the ratios of successive terms of the Fibonacci sequence (or any Fibonacci-like sequence), as originally shown by Kepler:[19]

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \varphi.$$

Therefore, if a Fibonacci number is divided by its immediate predecessor in the sequence, the quotient approximates φ; e.g., 987/610 ≈ 1.6180327868852. These approximations are alternately lower and higher than φ, and converge on φ as the Fibonacci numbers increase, and:

$$\sum_{n=1}^{\infty} |F(n)\varphi - F(n+1)| = \varphi.$$

More generally:

$$\lim_{n\to\infty} \frac{F(n+a)}{F(n)} = \varphi^a,$$

where above, the ratios of consecutive terms of the Fibonacci sequence, is a case when $a = 1$.

Furthermore, the successive powers of φ obey the Fibonacci recurrence:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}.$$

This identity allows any polynomial in φ to be reduced to a linear expression. For example:

$$\begin{aligned} 3\varphi^3 - 5\varphi^2 + 4 &= 3(\varphi^2 + \varphi) - 5\varphi^2 + 4 \\ &= 3[(\varphi + 1) + \varphi] - 5(\varphi + 1) + 4 \\ &= \varphi + 2 \approx 3.618. \end{aligned}$$

However, this is no special property of φ, because polynomials in any solution *x* to a quadratic equation can be reduced in an analogous manner, by applying:

$$x^2 = ax + b$$

for given coefficients *a*, *b* such that *x* satisfies the equation. Even more generally, any rational function (with rational coefficients) of the root of an irreducible *n*th-degree polynomial over the rationals can be reduced to a polynomial of degree *n* – 1. Phrased in terms of field theory, if α is a root of an irreducible *n*th-degree polynomial, then $\mathbb{Q}(\alpha)$ has degree *n* over $\mathbb{Q}$, with basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$.

Symmetries

The golden ratio and inverse golden ratio $\varphi_\pm = (1 \pm \sqrt{5})/2$ have a set of symmetries that preserve and interrelate them. They are both preserved by the fractional linear transformations $x, 1/(1-x), (x-1)/x,$ – this fact corresponds to the identity and the definition quadratic equation. Further, they are interchanged by the three maps $1/x, 1-x, x/(x-1)$ – they are reciprocals, symmetric about $1/2$, and (projectively) symmetric about 2.

More deeply, these maps form a subgroup of the modular group $\mathrm{PSL}(2, \mathbf{Z})$ isomorphic to the symmetric group on 3 letters, S_3, corresponding to the stabilizer of the set $\{0, 1, \infty\}$ of 3 standard points on the projective line, and the symmetries correspond to the quotient map $S_3 \to S_2$ – the subgroup $C_3 < S_3$ consisting of the 3-cycles and the identity $()(01\infty)(0\infty1)$ fixes the two numbers, while the 2-cycles interchange these, thus realizing the map.

Other properties

The golden ratio has the simplest expression (and slowest convergence) as a continued fraction expansion of any irrational number (see *Alternate forms* above). It is, for that reason, one of the worst cases of Lagrange's approximation theorem and it is an extremal case of the Hurwitz inequality for Diophantine approximations. This may be why angles close to the golden ratio often show up in phyllotaxis (the growth of plants).[71]

The defining quadratic polynomial and the conjugate relationship lead to decimal values that have their fractional part in common with φ:

$$\varphi^2 = \varphi + 1 = 2.618\ldots$$

$$\frac{1}{\varphi} = \varphi - 1 = 0.618\ldots.$$

The sequence of powers of φ contains these values 0.618…, 1.0, 1.618…, 2.618…; more generally, any power of φ is equal to the sum of the two immediately preceding powers:

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} = \varphi \cdot \mathrm{F}_n + \mathrm{F}_{n-1}\,.$$

As a result, one can easily decompose any power of φ into a multiple of φ and a constant. The multiple and the constant are always adjacent Fibonacci numbers. This leads to another property of the positive powers of φ:

If $\lfloor n/2 - 1 \rfloor = m$, then:

$$\varphi^n = \varphi^{n-1} + \varphi^{n-3} + \cdots + \varphi^{n-1-2m} + \varphi^{n-2-2m}$$

$$\varphi^n - \varphi^{n-1} = \varphi^{n-2}.$$

When the golden ratio is used as the base of a numeral system (see Golden ratio base, sometimes dubbed *phinary* or *φ-nary*), every integer has a terminating representation, despite φ being irrational, but every fraction has a non-terminating representation.

The golden ratio is a fundamental unit of the algebraic number field $\mathbb{Q}(\sqrt{5})$ and is a Pisot–Vijayaraghavan number.[72] In the field $\mathbb{Q}(\sqrt{5})$ we have $\varphi^n = \frac{L_n + F_n\sqrt{5}}{2}$, where L_n is the n-th Lucas number.

The golden ratio also appears in hyperbolic geometry, as the maximum distance from a point on one side of an ideal triangle to the closer of the other two sides: this distance, the side length of the equilateral triangle formed by the points of tangency of a circle inscribed within the ideal triangle, is 4 ln φ.[73]

Decimal expansion

The golden ratio's decimal expansion can be calculated directly from the expression

$$\varphi = \frac{1+\sqrt{5}}{2}$$

with √5 ≈ 2.2360679774997896964. The square root of 5 can be calculated with the Babylonian method, starting with an initial estimate such as $x\varphi = 2$ and iterating

$$x_{n+1} = \frac{(x_n + 5/x_n)}{2}$$

for $n = 1, 2, 3, \ldots$, until the difference between x_n and x_{n-1} becomes zero, to the desired number of digits.

The Babylonian algorithm for √5 is equivalent to Newton's method for solving the equation $x^2 - 5 = 0$. In its more general form, Newton's method can be applied directly to any algebraic equation, including the equation $x^2 - x - 1 = 0$ that defines the golden ratio. This gives an iteration that converges to the golden ratio itself,

$$x_{n+1} = \frac{x_n^2 + 1}{2x_n - 1},$$

for an appropriate initial estimate $x\varphi$ such as $x\varphi = 1$. A slightly faster method is to rewrite the equation as $x - 1 - 1/x = 0$, in which case the Newton iteration becomes

$$x_{n+1} = \frac{x_n^2 + 2x_n}{x_n^2 + 1}.$$

These iterations all converge quadratically; that is, each step roughly doubles the number of correct digits. The golden ratio is therefore relatively easy to compute with arbitrary precision. The time needed to compute n digits of the golden ratio is proportional to the time needed to divide two n-digit numbers. This is considerably faster than known algorithms for the transcendental numbers **π** and ***e***.

An easily programmed alternative using only integer arithmetic is to calculate two large consecutive Fibonacci numbers and divide them. The ratio of Fibonacci numbers F_{25001} and F_{25000}, each over 5000 digits, yields over 10,000 significant digits of the golden ratio.

The golden ratio *φ* has been calculated to an accuracy of several millions of decimal digits (sequence A001622 in OEIS). Alexis Irlande performed computations and verification of the first 17,000,000,000 digits.[74]

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